

Proof of Gravity and Yang-Mills Amplitude Relations

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ABSTRACT: Using BCFW on-shell recursion techniques, we prove a sequence of explicit n -point Kawai-Lewellen-Tye relations between gravity and Yang-Mills amplitudes at tree level.

KEYWORDS: Amplitudes, Field Theory, String Theory.

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1. Introduction

One of the most challenging problems in theoretical physics is the merging of quantum mechanics and gravity. Even at the most modest level of perturbative quantum gravity at tree level this is a daunting task. The Lagrangian of Einstein Gravity, although compact in its formal expression, explodes into an infinite series when expanded perturbatively around a given background. Calculations of scattering amplitudes directly by Feynman diagrams are virtually impossible, even at tree level, for more than a few external gravitons. We must therefore be more imaginative and try to look for alternatives. The famous set of Kawai-Lewellen-Tye (KLT) relations [1] provides just such an amazing alternative. The KLT-relations express, in a highly surprising manner, on-shell graviton amplitudes as ‘squares’ of on-shell color-ordered *Yang-Mills amplitudes*. If it were not for the existence of string theory, such remarkable relations would presumably never have been discovered. In string theory, where they were originally derived [1], they provide what is by now a textbook example of the relationship between open and closed string amplitudes. As such they are exact, and when expanded they hold to all orders in α' . Previously, the only known way to deduce the corresponding amplitude relations at the quantum field theory level was to take the limit $\alpha' \rightarrow 0$ of these stringy KLT-relations [2, 3].

Inspired by Witten [4], there has been really remarkable progress in Yang-Mills amplitude computations as reviewed in, *e.g.*, [5, 6]¹. Very recently, we have shown how the KLT-relations can be proven directly at the quantum field theory level, without recourse to string theory [8]. The method used in that paper was the on-shell BCFW recursion technique [9]. It relies only on general quantum field theory properties of the S -matrix and a certain convergence requirement on amplitudes at complex momenta.

¹For a review with a strong focus on KLT-relations see [7].

Much of the recent progress in the understanding of KLT-relations is due to the discovery by Bern, Carrasco and Johansson (BCJ) of a new set of gauge invariant relations between color-ordered Yang-Mills amplitudes [10]. While these relations (and their generalizations to include matter fields [11]) were at first conjectured on the basis of observed patterns in Yang-Mills amplitudes, it is interesting that the first proof [12] of their validity relied on first showing a more general set of relations in string theory and then taking the $\alpha' \rightarrow 0$ limit, see also ref. [13]. In this way string theory again plays a central role, and one can, using monodromy in string theory, easily prove in this way that the minimal basis of gauge theory amplitudes with n external legs grows only like $(n-3)!$ [12], and not like $(n-2)!$ as would be concluded on the basis of the field theory Kleiss-Kuijf relations [14, 15] alone. Quite recently both sets of relations have been proven by means of BCFW-recursion [16] (see also [17]), thus also there circumventing the need to go through string theory first.

The fact that BCJ-relations are central to KLT-relations and to the various rewritings of them becomes quite obvious in this paper. In this connection it is interesting that the first example of a BCJ-like relation is implicitly contained in ref. [1] by noting that alternative factorizations of closed string amplitudes into products of open string amplitudes yield different expressions. Equating the two, a new identity is inferred, and this is indeed one particular BCJ-relation. From our perspective today, this is not surprising since it is monodromy [12, 13] that is the mechanism behind the stringy BCJ-relations. However, our aim in this paper is to phrase the whole discussion in terms of quantum field theory concepts alone. It turns out that the one surviving property from string theory that we need to use is precisely just the information encoded in the field theory BCJ-relations.

The particular versions of the field theory KLT-relations that were proven in ref. [8] are not those commonly referred to before and which were conjectured in ref. [18]. The relations proven in [8] keep two external legs fixed while summing over permutations of the remaining. This is ideally suited for a proof based on on-shell recursion. The disadvantage is, as will be reviewed below, that they use a representation that contains an apparent singularity when all legs are on-shell. They therefore need to be supplemented with a suitable regularization. Once regulated, the on-shell field theory BCJ-relations can be used to remove the apparent singularity in a systematic manner. In this way, one ends up with relations that can be written in more familiar forms. While this is sufficient as a matter of principle, one would still like to have a corresponding direct proof of the conjectured KLT-formula for arbitrary n -point amplitudes that can be found in ref. [18]. In the course of another investigation dealing with a new set of gauge theory identities [19]² we uncovered a much wider sequence of explicit field theory KLT-relations that are all, via BCJ-relations, equivalent.³ The purpose of this paper is to prove these explicit field theory KLT-relations.

The plan of our paper is as follows. In section 2 we introduce the different versions of field theory KLT-relations we will prove by BCFW-recursion. We also discuss two very useful functions that form the backbone of KLT-relations in the field theory limit. It is due to the remarkable properties of these functions that we can so easily move between different formulations. In par-

²For the generalization to the full $\mathcal{N} = 4$ multiplet, see [20]. An interpretation of these identities in terms of vanishing gravity amplitudes with complex scalars has recently been given by Tye and Zhang [21].

³After the completion of [19] we discovered another paper [22] where an arbitrariness of the expression conjectured in [18] was noted. We believe, but have not checked in detail, that this arbitrariness is the same that we exploited in [19] and which we will prove explicitly below.

ticular, these functions encode the BCJ-relations and almost miraculously cancel poles so as to provide gravity amplitudes from products of two gauge theory amplitudes. In section 3 we give the explicit proofs. It is interesting that these proofs simplify greatly by having at our disposal several different versions of the KLT-relations. Since these different versions are related through BCJ-relations this shows how important these BCJ-relations are for establishing the field theory proof of KLT-relations. Section 4 contains our conclusions.

2. KLT-relations

We begin by presenting several different forms of the KLT-relations in the field theory limit. The first is the version presented and proven in ref. [8]. It keeps just two legs fixed in each gauge theory amplitude and sums over $(n - 2)!$ permutations. This expression has the advantage of having a high degree of manifest permutation symmetry, but the price one pays is that it contains, necessarily, many terms. As mentioned in the Introduction, it also has an apparent singularity and therefore requires regularization. Once regularized, one sees that the apparent pole is canceled by a corresponding term in the numerator. This KLT-form in which just two legs are kept fixed (rather than three, see below) is ideally suited for a proof using on-shell recursion in the form of BCFW-shifts, the two legs that are kept fixed naturally lending themselves to the needed momentum shifts.

Next we introduce a whole family of alternative KLT-relations that have less manifest permutation symmetry. They contain fewer terms and have no need of a regularization. We have already noted these alternative forms in a paper on new identities in gauge theories [19]. Here we will explicitly prove that they are all equivalent. Because they contain fewer terms, they might be regarded as more useful in practice. Among these new forms is also the originally conjectured n -point KLT-relation of [18]. It is thus a special case.

Although these more compact KLT-expressions are not ideally suited for a proof based on BCFW-recursion with just two legs shifted, we can nevertheless carry such a proof through. This will be shown below.

2.1 Two useful functions

Before presenting the KLT-relations let us for the sake of completeness introduce some useful definitions and the notation that will be used throughout the paper.

We begin with the following \mathcal{S} -function [8]

$$\mathcal{S}[i_1, \dots, i_k | j_1, \dots, j_k] = \prod_{t=1}^k \left(s_{i_t 1} + \sum_{q>t}^k \theta(i_t, i_q) s_{i_t i_q} \right), \quad (2.1)$$

where $\theta(i_t, i_q)$ is zero when the pair (i_t, i_q) has same ordering at both sets $\mathcal{I} \equiv \{i_1, \dots, i_k\}$ and $\mathcal{J} \equiv \{j_1, \dots, j_k\}$, but otherwise it is unity (we will everywhere use $s_{ij} \equiv (p_i + p_j)^2$ and $s_{i\dots k} \equiv (p_i + \dots + p_k)^2$). The function \mathcal{S} is nothing but a rewriting of the f -function defined in [18], now with a notation that we find more compelling. In this definition, the set \mathcal{J} is the reference ordering set, *i.e.*, this set provides the standard ordering. The set \mathcal{I} determines the given factor in the product by comparing with the set \mathcal{J} . It might be useful to illustrate the definition with a few examples

$$\mathcal{S}[2, 3, 4 | 2, 4, 3] = s_{21}(s_{31} + s_{34})s_{41}, \quad \mathcal{S}[2, 3, 4 | 4, 3, 2] = (s_{21} + s_{23} + s_{24})(s_{31} + s_{34})s_{41}.$$

Note that for each leg i there is a term s_{i1} . In other words, the momentum p_1 has been singled out and it plays a special role. When we have several functions \mathcal{S} with different choices of p_1 , we will write $\mathcal{S}[\mathcal{I}|\mathcal{J}]_{p_1}$ to avoid confusion.

One important property of the function \mathcal{S} is the reflection symmetry

$$\mathcal{S}[i_1, \dots, i_k | j_1, \dots, j_k] = \mathcal{S}[j_k, \dots, j_1 | i_k, \dots, i_1], \quad (2.2)$$

where we have exchanged the two sets and at the same time reversed the ordering of each set. The reason for this identity is clear. First we notice that if (x, y) at both sets \mathcal{I} and \mathcal{J} have the same (different) ordering, they have same (different) ordering after swapping the sets and reversing the order. If the ordering is the same for the pair (x, y) in both sets there is no corresponding s_{xy} . If the ordering is different for the pair (x, y) we have, before swapping, $\mathcal{S}[A, x, B, y, C | D, y, E, x, F]$. We thus have a factor $(s_{x1} + s_{xy} + \dots)$, while after the exchange we have $\mathcal{S}[F, x, E, y, D | C, y, B, x, A]$ and hence the same factor $(s_{x1} + s_{xy} + \dots)$.

It is also convenient to introduce a dual $\tilde{\mathcal{S}}$ -function

$$\tilde{\mathcal{S}}[i_1, \dots, i_k | j_1, \dots, j_k] \equiv \prod_{t=1}^k (s_{j_t n} + \sum_{q < t} \theta(j_q, j_t) s_{j_t j_q}), \quad (2.3)$$

where again $\theta(j_a, j_b)$ is zero if j_a sequentially comes before j_b in $\{i_1, \dots, i_k\}$, and otherwise it is unity. This dual form matches precisely the \tilde{f} -function defined in [18], again rewritten in a way that we find convenient. For this dual $\tilde{\mathcal{S}}$, we will think of the standard ordering as given by the set \mathcal{I} . It is worth emphasizing that the summation in (2.3) is over $q < t$, *i.e.*, over elements to the left of j_t in the set \mathcal{J} . As an example,

$$\tilde{\mathcal{S}}[2, 3, 4 | 4, 3, 2]_{p_5} = s_{45}(s_{35} + s_{34})(s_{25} + s_{23} + s_{24}).$$

And $\tilde{\mathcal{S}}$ has the same reflection symmetry as \mathcal{S}

$$\tilde{\mathcal{S}}[i_1, \dots, i_k | j_1, \dots, j_k] = \tilde{\mathcal{S}}[j_k, \dots, j_1 | i_k, \dots, i_1]. \quad (2.4)$$

The functions \mathcal{S} and $\tilde{\mathcal{S}}$ have several other useful properties. Considered as ‘operators’ they annihilate amplitudes in the following sense

$$\sum_{\alpha \in \mathcal{S}_{n-2}} \mathcal{S}[\alpha_{2,n-1} | j_2, \dots, j_{n-1}]_{p_1} A_n(n, \alpha_{2,n-1}, 1) = 0. \quad (2.5)$$

We will throughout use the shorthand notation: $\alpha_{2,n-1}$ for the ordering α of legs $2, 3, \dots, n-1$ in amplitudes. This property is in fact nothing but a rephrasing of BCJ-relations. To see this, let us first consider a six-point example with $\mathcal{J} = (2, 3, 4, 5)$. The sum of permutation $\alpha_{2,5}$ can be divided into permutations of the set $\{2, 3, 4\}$ plus 5 placed at all possible positions. Considering one particular ordering of $\{2, 3, 4\}$, for example the ordering $(3, 4, 2)$, we have the sum

$$\begin{aligned} & \mathcal{S}[3, 4, 2, 5 | 2, 3, 4, 5] A_6(6, 3, 4, 2, 5, 1) + \mathcal{S}[3, 4, 5, 2 | 2, 3, 4, 5] A_6(6, 3, 4, 5, 2, 1) \\ & + \mathcal{S}[3, 5, 4, 2 | 2, 3, 4, 5] A_6(6, 3, 5, 4, 2, 1) + \mathcal{S}[5, 3, 4, 2 | 2, 3, 4, 5] A_6(6, 5, 3, 4, 2, 1) \\ & = (s_{31} + s_{32})(s_{41} + s_{42}) s_{21} [s_{51} A_6(6, 3, 4, 2, 5, 1) + (s_{51} + s_{52}) A_6(6, 3, 4, 5, 2, 1) \\ & + (s_{51} + s_{52} + s_{54}) A_6(6, 3, 5, 4, 2, 1) + (s_{51} + s_{52} + s_{54} + s_{53}) A_6(6, 5, 3, 4, 2, 1)] , \end{aligned} \quad (2.6)$$

which indeed vanishes as a consequence of the BCJ-relation

$$0 = s_{51}A_6(6, 3, 4, 2, 5, 1) + (s_{51} + s_{52})A_6(6, 3, 4, 5, 2, 1) \\ + (s_{51} + s_{52} + s_{54})A_6(6, 3, 5, 4, 2, 1) + (s_{51} + s_{52} + s_{54} + s_{53})A_6(6, 5, 3, 4, 2, 1). \quad (2.7)$$

The above relation, *i.e.* eq. (2.7), is given in the form that we from now on will denote as a *fundamental* BCJ-relation. Note how a leg j (above $j = 5$) moves one leg to the left in each term and picks up an additional factor of s_{ji} (above s_{5i}), where i is the leg just passed.

With the above example at hand it is easy to see how this argument generalizes. We divide the sum of $\alpha_{2,n-1}$ into the sum of groups where all, except j_{n-1} , have fixed ordering, and then insert j_{n-1} at any place. For each group all factors from \mathcal{S} are the same except the factor contributing from j_{n-1} . Like in the above example this will provide us with a fundamental BCJ-relation and thus vanish.

There is clearly an analogous relation for $\tilde{\mathcal{S}}$:

$$\sum_{\gamma \in S_{n-2}} \tilde{\mathcal{S}}[i_2, \dots, i_{n-1} | \gamma_{2,n-1}]_{p_n} \tilde{A}_n(n, \gamma_{2,n-1}, 1) = 0. \quad (2.8)$$

2.2 New KLT forms

We are now ready to present the actual KLT-relations. The new form that was presented and proven in [8] is

$$M_n = (-1)^n \sum_{\gamma, \beta} \frac{\tilde{A}_n(n, \gamma_{2,n-1}, 1) \mathcal{S}[\gamma_{2,n-1} | \beta_{2,n-1}]_{p_1} A_n(1, \beta_{2,n-1}, n)}{s_{12\dots(n-1)}}. \quad (2.9)$$

It also has a dual form given by

$$M_n = (-1)^n \sum_{\beta, \gamma} \frac{A_n(1, \beta_{2,n-1}, n) \tilde{\mathcal{S}}[\beta_{2,n-1} | \gamma_{2,n-1}]_{p_n} \tilde{A}_n(n, \gamma_{2,n-1}, 1)}{s_{23\dots n}}, \quad (2.10)$$

where A_n and \tilde{A}_n are gauge theory amplitudes and M_n is the gravity amplitude. Although only eq. (2.9) was proven in [8], eq. (2.10) can obviously be proven in exactly the same manner and we will not repeat the proof here. As already discussed in [8], the above forms are singular when all legs are on-shell. They require regularization, which fortunately is easily implemented, see ref. [8] for details.

2.3 The general KLT expression

In terms of the \mathcal{S} -functions defined above the explicit n -point KLT-relation conjectured in [18] can be written as

$$M_n = (-1)^{n+1} \sum_{\sigma \in S_{n-3}} \sum_{\alpha \in S_{j-1}} \sum_{\beta \in S_{n-2-j}} A_n(1, \sigma_{2,j}, \sigma_{j+1,n-2}, n-1, n) \mathcal{S}[\alpha_{\sigma(2),\sigma(j)} | \sigma_{2,j}]_{p_1} \\ \times \tilde{\mathcal{S}}[\sigma_{j+1,n-2} | \beta_{\sigma(j+1),\sigma(n-2)}]_{p_{n-1}} \tilde{A}_n(\alpha_{\sigma(2),\sigma(j)}, 1, n-1, \beta_{\sigma(j+1),\sigma(n-2)}, n), \quad (2.11)$$

where $j = \lfloor n/2 \rfloor$ is a fixed number ($\lfloor x \rfloor$ is the integer value of x). However, by the use of BCJ-relations we will see that the formula (2.11) actually holds for arbitrary j . It therefore presents us with a whole family of equivalent KLT-relations [19].

To see this generalization, let us first start with an example of $n = 8$ before moving to the general n -point case. The term multiplying $A_8(1, (2, 3, 4), (5, 6), 7, 8)$ is given by

$$\sum_{\alpha \in S_3} \sum_{\beta \in S_2} \mathcal{S}[\alpha_{2,4}|2, 3, 4]_{p_1} \tilde{\mathcal{S}}[5, 6|\beta_{5,6}]_{p_7} \tilde{A}_8(\alpha_{2,4}, 1, 7, \beta_{5,6}, 8). \quad (2.12)$$

The sum of permutations $\alpha_{2,4}$ can be divided into groups of fixed permutation of $\tilde{\alpha}_{2,3}$ plus all possible insertions of 4. Thus, with fixed permutation β , for example $\beta_{5,6} = (6, 5)$, and fixed permutation $\tilde{\alpha}$, for example $\tilde{\alpha}_{2,3} = (3, 2)$, the sum is given by

$$\begin{aligned} & \mathcal{S}[3, 2, 4|2, 3, 4]_{p_1} \tilde{\mathcal{S}}[5, 6|6, 5]_{p_7} \tilde{A}_8(3, 2, 4, 1, 7, 6, 5, 8) \\ & + \mathcal{S}[3, 4, 2|2, 3, 4]_{p_1} \tilde{\mathcal{S}}[5, 6|6, 5]_{p_7} \tilde{A}_8(3, 4, 2, 1, 7, 6, 5, 8) \\ & + \mathcal{S}[4, 3, 2|2, 3, 4]_{p_1} \tilde{\mathcal{S}}[5, 6|6, 5]_{p_7} \tilde{A}_8(4, 3, 2, 1, 7, 6, 5, 8) \\ & = s_{67}(s_{57} + s_{56})(s_{31} + s_{32})s_{21} \left[s_{41} \tilde{A}_8(3, 2, 4, 1, 7, 6, 5, 8) \right. \\ & \quad \left. + (s_{41} + s_{42}) \tilde{A}_8(3, 4, 2, 1, 7, 6, 5, 8) + (s_{41} + s_{42} + s_{43}) \tilde{A}_8(4, 3, 2, 1, 7, 6, 5, 8) \right]. \end{aligned} \quad (2.13)$$

We now use the fundamental BCJ-relation

$$\begin{aligned} 0 &= s_{41} \tilde{A}_8(3, 2, 4, 1, 7, 6, 5, 8) + (s_{41} + s_{42}) \tilde{A}_8(3, 4, 2, 1, 7, 6, 5, 8) \\ & \quad + (s_{41} + s_{42} + s_{43}) \tilde{A}_8(4, 3, 2, 1, 7, 6, 5, 8) + (s_{41} + s_{42} + s_{43} + s_{48}) \tilde{A}_8(3, 2, 1, 7, 6, 5, 4, 8) \\ & \quad + (s_{41} + s_{42} + s_{43} + s_{48} + s_{45}) \tilde{A}_8(3, 2, 1, 7, 6, 4, 5, 8) \\ & \quad + (s_{41} + s_{42} + s_{43} + s_{48} + s_{45} + s_{46}) \tilde{A}_8(3, 2, 1, 7, 4, 6, 5, 8), \end{aligned} \quad (2.14)$$

where 4 is the leg being moved to the left. However, by momentum conservation we also have

$$\begin{aligned} s_{41} + s_{42} + s_{43} + s_{48} &= -(s_{47} + s_{45} + s_{46}), & s_{41} + s_{42} + s_{43} + s_{48} + s_{45} &= -(s_{47} + s_{46}), \\ s_{41} + s_{42} + s_{43} + s_{48} + s_{45} + s_{46} &= -s_{47}, \end{aligned} \quad (2.15)$$

and eq. (2.13) takes the form

$$\begin{aligned} & s_{67}(s_{57} + s_{56})(s_{31} + s_{32})s_{21} \left[(s_{47} + s_{45} + s_{46}) \tilde{A}_8(3, 2, 1, 7, 6, 5, 4, 8) \right. \\ & \quad \left. + (s_{47} + s_{46}) \tilde{A}_8(3, 2, 1, 7, 6, 4, 5, 8) + s_{47} \tilde{A}_8(3, 2, 1, 7, 4, 6, 5, 8) \right] \\ & = \mathcal{S}[3, 2|2, 3]_{p_1} \left[\tilde{\mathcal{S}}[4, 5, 6|6, 5, 4]_{p_7} \tilde{A}_8(3, 2, 1, 7, 6, 5, 4, 8) + \tilde{\mathcal{S}}[4, 5, 6|6, 4, 5]_{p_7} \tilde{A}_8(3, 2, 1, 7, 6, 4, 5, 8) \right. \\ & \quad \left. + \tilde{\mathcal{S}}[4, 5, 6|4, 6, 5]_{p_7} \tilde{A}_8(3, 2, 1, 7, 4, 6, 5, 8) \right]. \end{aligned} \quad (2.16)$$

Summing up all possible permutations $\tilde{\alpha}, \beta$ we find that the term multiplying

$$A_8(1, (2, 3, 4), (5, 6), 7, 8) = A_8(1, (2, 3), (4, 5, 6), 7, 8)$$

is given by

$$\sum_{\alpha \in S_2} \sum_{\beta \in S_3} \mathcal{S}[\alpha_{2,3}|2,3]_{p_1} \tilde{\mathcal{S}}[4,5,6|\beta_{4,6}]_{p_7} \tilde{A}_8(\alpha_{2,3}, 1, 7, \beta_{4,6}, 8), \quad (2.17)$$

where leg 4 has been moved from the set of \mathcal{S} into the set of $\tilde{\mathcal{S}}$. This can be done for any of the σ permutations and thereby give us the eight-point relation in the form of eq. (2.11) but with j shifted down by one. Repeating this argument we can get to any of the j -value forms we want.

We can now follow the same procedure for general n and obtain the following relation

$$\begin{aligned} & \sum_{\alpha, \beta} \mathcal{S}[\alpha_{i_2, i_j}|i_2, \dots, i_j]_{p_1} \tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2}|\beta_{i_{j+1}, i_{n-2}}]_{p_{n-1}} \tilde{A}_n(\alpha_{i_2, i_j}, 1, n-1, \beta_{i_{j+1}, i_{n-2}}, n) \\ &= \sum_{\alpha', \beta'} \mathcal{S}[\alpha'_{i_2, i_{j-1}}|i_2, \dots, i_{j-1}]_{p_1} \tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2}|\beta'_{i_j, i_{n-2}}]_{p_{n-1}} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, \beta'_{i_j, i_{n-2}}, n), \end{aligned} \quad (2.18)$$

from which the equivalence of eq. (2.11) for any j -value follows. The general proof of eq. (2.18) is relegated to Appendix A.

One interesting application of this result is that we can shift j all the way to make the left- or right-hand part empty, *i.e.* we can choose $j = 1$ or $j = n - 2$. Then we get the following two expressions [19]

$$M_n = (-1)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} \tilde{A}_n(n-1, n, \tilde{\sigma}_{2, n-2}, 1) \mathcal{S}[\tilde{\sigma}_{2, n-2}|\sigma_{2, n-2}]_{p_1} A_n(1, \sigma_{2, n-2}, n-1, n), \quad (2.19)$$

and

$$M_n = (-1)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} A_n(1, \sigma_{2, n-2}, n-1, n) \tilde{\mathcal{S}}[\sigma_{2, n-2}|\tilde{\sigma}_{2, n-2}]_{p_{n-1}} \tilde{A}_n(1, n-1, \tilde{\sigma}_{2, n-2}, n), \quad (2.20)$$

which are exactly related to each other by

$$\sum_{\tilde{\sigma}} \tilde{A}_n(n-1, n, \tilde{\sigma}_{2, n-2}, 1) \mathcal{S}[\tilde{\sigma}_{2, n-2}|\sigma_{2, n-2}]_{p_1} = \sum_{\tilde{\sigma}} \tilde{A}_n(1, n-1, \tilde{\sigma}_{2, n-2}, n) \tilde{\mathcal{S}}[\sigma_{2, n-2}|\tilde{\sigma}_{2, n-2}]_{p_{n-1}}, \quad (2.21)$$

following from repeated use of eq. (2.18).

We will call the latter form, *i.e.* eq. (2.20), the *dual form* of (2.19). It is important to stress again that all these forms are completely equivalent. As we have explained above and prove in detail in Appendix A, they are related to each other by use of BCJ-relations through eq. (2.18).

3. Field theory proof of KLT-relations

We now turn to the field theory proof of the whole family of KLT-expressions introduced above. As we will see, the $(n-2)!$ symmetric form will be an important ingredient even for the proof of the relations that are only manifestly $(n-3)!$ symmetric.

Since all forms in the family of eq. (2.11) (*i.e.* with arbitrary j) are related to each other by BCJ-relations, we are free to choose any of the versions we find most convenient. Proving one version clearly proves them all.

As in [8] we will give an induction proof by means of BCFW-recursion. Specifically, we will make a $(1, n-1)$ BCFW-shift and consider the integral

$$0 = \oint \frac{dz}{z} M_n(z) = M_n(0) + \sum (\text{residues for } z \neq 0). \quad (3.1)$$

Evidently, if there are boundary terms to this contour integral, they are not included here. It has been shown in refs. [23, 24] that the fall-off at $z \rightarrow \infty$ of the graviton amplitude $M_n(z)$ is even stronger than one could naively have guessed, and goes as $1/z^2$. Our aim is to show that the sum of residues exactly gives us the BCFW-expansion of the n -point gravity amplitude through use of only lower-point KLT relations. It is easy to see that the relations are indeed satisfied for low-point relations.

To simplify the expressions as much as possible we will mainly use the simple versions given by eq. (2.19) and (2.20). For convenience we discard the overall sign $(-1)^{n+1}$, which is readily reinstated into the proof.

We start by considering all the residues coming from poles of the form $s_{12\dots k}$ (we can compute the residue of the pole $s_{\widehat{12\dots k}}$ as $\sim \lim_{z \rightarrow z_{12\dots k}} [s_{\widehat{12\dots k}}(z) M_n(z)] / s_{12\dots k}$, where $z_{12\dots k}$ is the z -value that makes $s_{\widehat{12\dots k}}$ go on-shell).

There are three cases

- (A-1) The pole appears only in \widetilde{A}_n .
- (A-2) The pole appears only in A_n .
- (B) The pole appears both in \widetilde{A}_n and A_n .

Starting with (A-1) we see that the contributing terms from eq. (2.19) will be of the form

$$\sum_{\sigma, \widetilde{\sigma}, \alpha} \widetilde{A}_n(\widehat{n-1}, n, \widetilde{\sigma}_{k+1, n-2}, \alpha_{2, k}, \widehat{1}) \mathcal{S}[\widetilde{\sigma}_{k+1, n-2} \alpha_{2, k} | \sigma_{2, n-2}]_{\widehat{p}_1} A_n(\widehat{1}, \sigma_{2, n-2}, \widehat{n-1}, n), \quad (3.2)$$

and we therefore get the residue

$$\begin{aligned} \sum_{\sigma, \widetilde{\sigma}, \alpha} \frac{\sum_h \widetilde{A}(\widehat{n-1}, n, \widetilde{\sigma}_{k+1, n-2}, \widehat{P}^h) \widetilde{A}(-\widehat{P}^{-h}, \alpha_{2, k}, \widehat{1})}{s_{12\dots k}} \mathcal{S}[\widetilde{\sigma}_{k+1, n-2} \alpha_{2, k} | \sigma_{2, n-2}]_{\widehat{p}_1} \\ \times A_n(\widehat{1}, \sigma_{2, n-2}, \widehat{n-1}, n). \end{aligned} \quad (3.3)$$

We now use the important factorization property (which follows from the definition of \mathcal{S})

$$\mathcal{S}[\widetilde{\sigma}_{k+1, n-2} \alpha_{2, k} | \sigma_{2, n-2}]_{\widehat{p}_1} = \mathcal{S}[\alpha_{2, k} | \rho_{2, k}]_{\widehat{p}_1} \times (\text{a factor independent of } \alpha), \quad (3.4)$$

where ρ denotes the relative ordering of leg $2, 3, \dots, k$ in the set σ . It follows that this contribution contains an expression like

$$\sim \sum_{\alpha} \tilde{A}(-\hat{P}^{-h}, \alpha_{2,k}, \hat{1}) \mathcal{S}[\alpha_{2,k} | \rho_{2,k}]_{\hat{p}_1} = 0, \quad (3.5)$$

which, as we have seen above, vanishes due to BCJ-relations. We therefore conclude that contributions from (A-1) vanish altogether.

With a similar argument we note that contributions from (A-2) also vanish.

Moving on to (B) the contributing terms in eq. (2.19) take the form

$$\begin{aligned} \sum_{\sigma, \tilde{\sigma}, \alpha, \beta} \tilde{A}_n(\widehat{n-1}, n, \tilde{\sigma}_{k+1, n-2}, \alpha_{2,k}, \hat{1}) \mathcal{S}[\tilde{\sigma}_{k+1, n-2} \alpha_{2,k} | \beta_{2,k} \sigma_{k+1, n-2}]_{\hat{p}_1} \\ \times A_n(\hat{1}, \beta_{2,k}, \sigma_{k+1, n-2}, \widehat{n-1}, n). \end{aligned} \quad (3.6)$$

Using the factorization property

$$\mathcal{S}[\tilde{\sigma}_{k+1, n-2} \alpha_{2,k} | \beta_{2,k} \sigma_{k+1, n-2}]_{\hat{p}_1} = \mathcal{S}[\alpha_{2,k} | \beta_{2,k}]_{\hat{p}_1} \times \mathcal{S}[\tilde{\sigma}_{k+1, n-2} | \sigma_{k+1, n-2}]_{\hat{p}}, \quad (3.7)$$

the residue for $s_{12\dots k}$ can be written as

$$\begin{aligned} \frac{1}{s_{12\dots k}} \sum_h \sum_{\alpha, \beta} \frac{\tilde{A}(-\hat{P}^h, \alpha_{2,k}, \hat{1}) \mathcal{S}[\alpha_{2,k} | \beta_{2,k}]_{\hat{p}_1} A(\hat{1}, \beta_{2,k}, -\hat{P}^h)}{s_{\hat{1}2\dots k}} \\ \times \sum_{\sigma, \tilde{\sigma}} \tilde{A}(\widehat{n-1}, n, \tilde{\sigma}_{k+1, n-2}, \hat{P}^{-h}) \mathcal{S}[\tilde{\sigma}_{k+1, n-2} | \sigma_{k+1, n-2}]_{\hat{p}} A(\hat{P}^{-h}, \sigma_{k+1, n-2}, \widehat{n-1}, n), \end{aligned} \quad (3.8)$$

where we have used the vanishing of all the mixed-helicity contributions [8, 19].

Remarkably, the sum over α and β is precisely the singular KLT-form of ref. [8]. As shown in that paper, it is equal to $M_{k+1}(\hat{1}, 2, \dots, k, -\hat{P}^h)$. Moreover, the sum over σ and $\tilde{\sigma}$ is an $n-k+1$ point version of eq. (2.19) and hence, by induction, equal to $M_{n-k+1}(k+1, \dots, \hat{P}^{-h})$, *i.e.*

$$\sum_h \frac{M_{k+1}(\hat{1}, 2, \dots, k, -\hat{P}^h) M_{n-k+1}(k+1, \dots, \hat{P}^{-h})}{s_{12\dots k}}. \quad (3.9)$$

From this we conclude that we get the correct BCFW-contributions to the n -point gravity amplitude for all poles of the $s_{12\dots k}$ kind, and by the manifest $(n-3)!$ symmetry also for all poles related to these by permutation of leg $2, 3, \dots, n-2$.

However, this is not the end of the proof. Because of the only $(n-3)!$ manifest symmetry we also need to explicitly consider pole contributions involving leg 1 and n , *i.e.* poles of the form $s_{12\dots n\dots k} = s_{k+1\dots n-1}$. To investigate these contributions we will use the dual form, eq. (2.20). This is allowed since we have seen that eq. (2.19) and (2.20) are just two different ways of writing the same quantity.

Again we start with (A-1), but this time considering the $s_{12\dots n\dots k}$ pole and using eq. (2.20) instead of the equivalent eq. (2.19). The residue takes the form

$$\sum_{\sigma, \tilde{\sigma}, \alpha} A_n(\widehat{1}, \sigma_{2,n-2}, \widehat{n-1}, n) \tilde{\mathcal{S}}[\sigma_{2,n-2} | \tilde{\sigma}_{k+1,n-2} \alpha_{2,k}]_{\widehat{p}_{n-1}} \times \sum_h \frac{\tilde{A}(\widehat{n-1}, \tilde{\sigma}_{k+1,n-2}, \widehat{P}^h) \tilde{A}(-\widehat{P}^{-h}, \alpha_{2,k}, n, \widehat{1})}{s_{12\dots n\dots k}}. \quad (3.10)$$

Using the factorization property

$$\tilde{\mathcal{S}}[\sigma_{2,n-2} | \tilde{\sigma}_{k+1,n-2} \alpha_{2,k}]_{\widehat{p}_{n-1}} = \tilde{\mathcal{S}}[\rho_{k+1,n-2} | \tilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}} \times (\text{a factor independent of } \tilde{\sigma}), \quad (3.11)$$

where ρ denotes the relative ordering of leg $k+1, \dots, n-2$ in set σ , we once again see that the contribution contains a factor of

$$\sim \sum_{\tilde{\sigma}} \tilde{A}(\widehat{n-1}, \tilde{\sigma}_{k+1,n-2}, \widehat{P}^h) \tilde{\mathcal{S}}[\rho_{k+1,n-2} | \tilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}} = 0, \quad (3.12)$$

that vanishes due to BCJ-relations.

Similar arguments apply to (A-2) for the $s_{12\dots n\dots k}$ pole.

Now we consider (B) for a $s_{12\dots n\dots k}$ pole. In this case the contributing terms of eq. (2.20) have the form

$$\sum_{\sigma, \tilde{\sigma}, \alpha, \beta} A_n(\widehat{1}, \beta_{2,k}, \sigma_{k+1,n-2}, \widehat{n-1}, n) \tilde{\mathcal{S}}[\beta_{2,k} \sigma_{k+1,n-2} | \tilde{\sigma}_{k+1,n-2} \alpha_{2,k}]_{\widehat{p}_{n-1}} \times \tilde{A}_n(\widehat{1}, \widehat{n-1}, \tilde{\sigma}_{k+1,n-2}, \alpha_{2,k}, n), \quad (3.13)$$

and $\tilde{\mathcal{S}}$ satisfies the factorization property

$$\tilde{\mathcal{S}}[\beta_{2,k} \sigma_{k+1,n-2} | \tilde{\sigma}_{k+1,n-2} \alpha_{2,k}]_{\widehat{p}_{n-1}} = \tilde{\mathcal{S}}[\sigma_{k+1,n-2} | \tilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}} \tilde{\mathcal{S}}[\beta_{2,k} | \alpha_{2,k}]_{\widehat{P}}. \quad (3.14)$$

Hence the residue can be written

$$\frac{1}{s_{12\dots n\dots k}} \sum_h \sum_{\alpha, \beta} A(\widehat{1}, \beta_{2,k}, \widehat{P}^h, n) \tilde{\mathcal{S}}[\beta_{2,k} | \alpha_{2,k}]_{\widehat{P}} \tilde{A}(\widehat{1}, \widehat{P}^h, \alpha_{2,k}, n) \times \sum_{\tilde{\sigma}, \sigma} \frac{A(-\widehat{P}^{-h}, \sigma_{k+1,n-2}, \widehat{n-1}) \tilde{\mathcal{S}}[\sigma_{k+1,n-2} | \tilde{\sigma}_{k+1,n-2}]_{\widehat{p}_{n-1}} \tilde{A}(\widehat{n-1}, \tilde{\sigma}_{k+1,n-2}, -\widehat{P}^{-h})}{s_{k+1\dots n-1}}, \quad (3.15)$$

where we have used $s_{\widehat{12\dots n\dots k}} = s_{k+1\dots \widehat{n-1}}$, and the vanishing of mixed-helicity contributions. We see that the first term is just a lower-point version of eq. (2.20), and the second term the singular dual KLT-form, *i.e.*

$$\sum_h \frac{M_{k+2}(\widehat{1}, 2, \dots, n, \widehat{P}^h) M_{n-k}(k+1, \dots, \widehat{n-1}, -\widehat{P}^{-h})}{s_{12\dots n\dots k}}. \quad (3.16)$$

We hereby see that we once again obtain the correct BCFW expansion for all $s_{12\dots n\dots k}$ poles, and by the symmetry of leg $2, \dots, n-2$ also for all poles related to these by a permutation.

The above analysis covers all residues for the n -point KLT-relation, which we see give the full BCFW-expansion for the n -point gravity amplitude. We stress that we have *not* assumed any permutation symmetries beyond that which are already manifest in the expressions. This therefore also constitutes a proof that these KLT-expression are fully symmetric under permutations since the gravity amplitudes have this property. It would be most tedious to prove this fact by more direct means.

4. Conclusion

We have provided a field theory proof by induction of a whole new class of KLT-expressions that are all equivalent to the particular case conjectured in ref. [18]. Only general properties of the S -matrix in quantum field theory [25] have been assumed.

As a byproduct of our study, we have demonstrated the validity of various rewritings of KLT-relations that we think will be of use. The two forms given in (2.19) and (2.20) and which have manifest $(n-3)!$ permutation symmetry are of particular interest. These two expressions are very similar to the new KLT-relations (2.9) that were presented in [8] and which have the larger $(n-2)!$ permutation symmetry. It would be of interest to establish directly, by algebraic means, the equivalence of these two forms by use of BCJ-relations. What we have shown in this paper is that they *are* identical since they both equal the n -point graviton amplitude. It is a highly non-trivial task to establish directly the relation between the $(n-2)!$ symmetric form and the $(n-3)!$ symmetric form. In principle, we know that we need only to use BCJ-relations to establish the connection, but the details are not simple.

One striking aspect of our proof is that it requires the simultaneous use of a variety of different KLT-expressions, and particular also the apparently singular version that was proven in ref. [8] is required. Because these are all connected via BCJ-relations, it is understandable why a field theory proof had to wait until these relations had been established. Another crucial ingredient is the use of BCFW-recursion. This shows how powerful such a recursive technique can be, not just for explicit computation of amplitudes, but also in a more abstract sense since they lend themselves easily to proofs based on induction.

There are various open ends that one would like to understand better. It would be very interesting to derive (2.9) from string theory. Unlike the case of [1], this time one should fix only two closed string vertices V_1, V_n . Presumably the volume of the remaining gauge symmetry accounts, in that framework and in the field theory limit, for the naively infinite $1/s_{12\dots(n-1)}$ that needs to be regularized by taking leg n off-shell. This picture seems intuitively reasonable, but it should be confirmed and the details need to be figured out. Having the manifest $(n-2)!$ symmetric form, it is curious to ask if one could find manifest $(n-1)!$ symmetric or even $n!$ symmetric forms as well. Another formulation of KLT-relations has also been investigated through a study of residues of poles [26, 27] and it would be interesting to consider our new more general forms of KLT-relations in this context.

A Lagrangian prescription for KLT-factorization is still lacking, although there has been progress in such a direction [28]. It would be interesting to investigate this further, possibly using

the methods of [29] now that it is clear how KLT-relations can be completely separated from a string theory origin. There are also alternative KLT-relations involving matter particles that are most easily derived from the heterotic string [30] and such relations have been shown explicitly to hold also for the corresponding effective field theories, expanded perturbatively in α' [31].

Another interesting problem is the following. By direct calculation and by using BCFW-recursion relations starting from $M_3 = A_3 \tilde{A}_3$, we find $M_4 = -s_{12} A(1, 2, 3, 4) \tilde{A}(1, 2, 4, 3)$ after use of a BCJ-relation. As we have stressed, the factor s_{12} is important to cancel the double pole from the product $A(1, 2, 3, 4) \tilde{A}(1, 2, 4, 3)$. In fact, for the general n , a crucial role of the function \mathcal{S} is to miraculously get rid of all double poles. It is natural to ask if we can uniquely fix the function \mathcal{S} from such a requirement. The answer to this question may be positive. Let us assume the result of n , i.e., we have

$$M_n = (-)^{n+1} \sum_{\beta, \alpha} \tilde{A}_n(n-1, n, \alpha_{2,n-2}, P) \mathcal{S}[\alpha|\beta]_P A_n(P, \beta_{2,n-2}, n-1, n), \quad (4.1)$$

where we have used P for particle p_1 . Going from n to $(n+1)$ we can consider P as the combination of two particles p_1, \tilde{p}_1 . We know that for the two-particle channel, we should have

$$\begin{aligned} M_{n+1} &\sim \sum_{\beta, \alpha} \tilde{A}_n(n-1, n, \alpha_{2,n-2}, \tilde{p}_1, p_1) \mathcal{S}[\alpha|\beta]_P s_{p_1 \tilde{p}_1} A_n(p_1, \tilde{p}_1, \beta_{2,n-2}, n-1, n) \\ &\sim \sum_{\beta, \alpha} \tilde{A}_n(n-1, n, \alpha_{2,n-2}, \tilde{p}_1, p_1) \mathcal{S}[\alpha, \tilde{p}_1 | \tilde{p}_1, \beta]_{p_1} A_n(p_1, \tilde{p}_1, \beta_{2,n-2}, n-1, n). \end{aligned} \quad (4.2)$$

To have full $(n-2)!$ permutation symmetry, we need to sum over all permutations between j and \tilde{p}_1 . It is then quite natural to extend

$$\sum_{\beta, \alpha} \mathcal{S}[\alpha, \tilde{p}_1 | \tilde{p}_1, \beta]_P \rightarrow \sum_{\tilde{\beta}, \tilde{\alpha}} \mathcal{S}[\tilde{\alpha}_{p_1, 2, \dots, n-2} | \tilde{\beta}_{\tilde{p}_1, 2, \dots, n-2}]. \quad (4.3)$$

By this line argument, it might be that one can recursively construct the general n -point KLT-relations by requiring correct single-pole structures in addition to correct multi-particle channels, a consistency condition that has also been discussed in ref. [18].

There are many other directions one can follow from here. It might be interesting to take a fresh look at string-based techniques such as those discussed in ref. [32]. We believe that it could be useful to consider our relations from string theory along the line of [33]. An obvious question is how much can be generalized to the loop level. Another interesting point is to see how the Grassmanian program initiated in [34] can be generalized to the $\mathcal{N} = 8$ theory, perhaps by using results discussed here. There will certainly be consistency requirements that must be met, and it would be quite amazing to see our different versions of KLT-relations emerge in that context.

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A. The shifting-formula for j

Here we provide a detailed proof of eq. (2.18). We begin with the following rewriting

$$\begin{aligned}
& \sum_{\alpha} \mathcal{S}[\alpha_{i_2, i_j} | i_2, \dots, i_j]_{p_1} \tilde{A}_n(\alpha_{i_2, i_j}, 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \\
&= \sum_{\alpha'} \left[\mathcal{S}[\alpha'_{i_2, i_{j-1}}, i_j | i_2, \dots, i_j]_{p_1} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, i_j, 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \right. \\
&+ \mathcal{S}[\alpha'_{i_2, i_{j-2}}, i_j, \alpha'(i_{j-1}) | i_2, \dots, i_j]_{p_1} \tilde{A}_n(\alpha'_{i_2, i_{j-2}}, i_j, \alpha'(i_{j-1}), 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \\
&+ \dots + \mathcal{S}[i_j, \alpha'_{i_2, i_{j-1}} | i_2, \dots, i_j]_{p_1} \tilde{A}_n(i_j, \alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \left. \right] \\
&= \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2, i_{j-1}} | i_2, \dots, i_{j-1}]_{p_1} \times \left[s_{1j} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, i_j, 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \right. \\
&+ (s_{1j} + s_{j\alpha'(j-1)}) \tilde{A}_n(\alpha'_{i_2, i_{j-2}}, i_j, \alpha'(i_{j-1}), 1, n-1, i_{j+1}, \dots, i_{n-2}, n) + \dots \\
&+ (s_{1j} + s_{j\alpha'(1)} + s_{j\alpha'(2)} + \dots + s_{j\alpha'(j-1)}) \tilde{A}_n(i_j, \alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \left. \right]. \tag{A.1}
\end{aligned}$$

Using the fundamental BCJ-relation on the expression inside $[\dots]$ as well as momentum conservation, we get

$$\begin{aligned}
& \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2, i_{j-1}} | i_2, \dots, i_{j-1}]_{p_1} \\
& \times \left[(s_{j(n-1)} + s_{j(j+1)} + s_{j(j+2)} + \dots + s_{j(n-2)}) \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-2}, i_j, n) \right. \\
&+ (s_{j(n-1)} + s_{j(j+1)} + s_{j(j+2)} + \dots + s_{j(n-3)}) \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-3}, i_j, i_{n-2}, n) + \dots \\
&+ s_{j(n-1)} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_j, i_{j+1}, \dots, i_{n-2}, n) \left. \right] \\
&= \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2, i_{j-1}} | i_2, \dots, i_{j-1}]_{p_1} \\
& \times \left[\frac{\tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-2}, i_j]_{p_{n-1}}}{\tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-2}]_{p_{n-1}}} \tilde{A}_n(\alpha'_{i_2, \dots, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-2}, i_j, n) \right. \\
&+ \frac{\tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-3}, i_j, i_{n-2}]_{p_{n-1}}}{\tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-2}]_{p_{n-1}}} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-3}, i_j, i_{n-2}, n) + \dots \\
&+ \frac{\tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | i_j, i_{j+1}, \dots, i_{n-2}]_{p_{n-1}}}{\tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-2}]_{p_{n-1}}} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_j, i_{j+1}, \dots, i_{n-2}, n) \left. \right], \tag{A.2}
\end{aligned}$$

i.e. by multiplying with $\tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2}|i_{j+1}, \dots, i_{n-2}]_{p_{n-1}}$ on both sides, we get the relation

$$\begin{aligned}
& \sum_{\alpha} \mathcal{S}[\alpha_{i_2, i_j} | i_2, \dots, i_j]_{p_1} \tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-2}]_{p_{n-1}} \tilde{A}_n(\alpha_{i_2, i_j}, 1, n-1, i_{j+1}, \dots, i_{n-2}, n) \\
&= \sum_{\alpha'} \mathcal{S}[\alpha'_{i_2, i_{j-1}} | i_2, \dots, i_{j-1}]_{p_1} \\
&\times \left[\tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-2}, i_j]_{p_{n-1}} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-2}, i_j, n) \right. \\
&+ \tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | i_{j+1}, \dots, i_{n-3}, i_j, i_{n-2}]_{p_{n-1}} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_{j+1}, \dots, i_{n-3}, i_j, i_{n-2}, n) + \dots \\
&\left. + \tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | i_j, i_{j+1}, \dots, i_{n-2}]_{p_{n-1}} \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, i_j, i_{j+1}, \dots, i_{n-2}, n) \right]. \tag{A.3}
\end{aligned}$$

We can now add all the permutations of $\text{leg } j+1, \dots, n-2$. On the left-hand side we get

$$\begin{aligned}
& \sum_{\alpha, \beta} \mathcal{S}[\alpha_{i_2, i_j} | i_2, \dots, i_j]_{p_1} \tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2} | \beta_{i_{j+1}, i_{n-2}}]_{p_{n-1}} \\
& \times \tilde{A}_n(\alpha_{i_2, i_j}, 1, n-1, \beta_{i_{j+1}, i_{n-2}}, n), \tag{A.4}
\end{aligned}$$

and on the right-hand side, since we have all permutations of i_j with the $\{j+1, \dots, n-2\}$ set, we can write it as all permutations of $\{i_j, i_{j+1}, \dots, i_{n-2}\}$

$$\begin{aligned}
& \sum_{\alpha', \beta'} \mathcal{S}[\alpha'_{i_2, i_{j-1}} | i_2, \dots, i_{j-1}]_{p_1} \tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | \beta'_{i_j, i_{n-2}}]_{p_{n-1}} \\
& \times \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, \beta'_{i_j, i_{n-2}}, n), \tag{A.5}
\end{aligned}$$

and we finally get the relation

$$\begin{aligned}
& \sum_{\alpha, \beta} \mathcal{S}[\alpha_{i_2, i_j} | i_2, \dots, i_j]_{p_1} \tilde{\mathcal{S}}[i_{j+1}, \dots, i_{n-2} | \beta_{i_{j+1}, i_{n-2}}]_{p_{n-1}} \\
& \times \tilde{A}_n(\alpha_{i_2, i_j}, 1, n-1, \beta_{i_{j+1}, i_{n-2}}, n) \\
&= \sum_{\alpha', \beta'} \mathcal{S}[\alpha'_{i_2, i_{j-1}} | i_2, \dots, i_{j-1}]_{p_1} \tilde{\mathcal{S}}[i_j, i_{j+1}, \dots, i_{n-2} | \beta'_{i_j, i_{n-2}}]_{p_{n-1}} \\
& \times \tilde{A}_n(\alpha'_{i_2, i_{j-1}}, 1, n-1, \beta'_{i_j, i_{n-2}}, n). \tag{A.6}
\end{aligned}$$

This ends our proof of eq. (2.18). This then also proves the j -independence of eq. (2.11).

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